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Nonlinear controller design for a helicopter with an external slung load system

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ABSTRACT

This paper describes a nonlinear deterministic controller design for a helicopter with an underslung external load system. A robust control approach is considered for the control law development. The system is considered as a cascade connection of an uncertain nonlinear system. The controller is designed to ensure the stabilization of the helicopter system and the positioning of the underslung load at hover condition. Control analysis and numerical results show that the proposed controller is able to locate the load at the specified position or its neighbourhood.

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Nonlinear controller; stability analysis; mathematical model; dynamic system

1. Introduction

The interest in designing a feedback controller for a helicopter by means of the nonlinear control strategy has nowadays gained considerable attention from several researchers (see, e.g. Avila-Vilchis, Brogliato, Dzul, & Lozano, 2003; Isidori, Marconi, & Serrani, 2003). Avila-Vilchis et al. (2003) present a nonlinear control strategy for a reduced-order model of a helicopter. Due to the complexity of the helicopter model and dynamics, it is very difficult to develop a nonlinear control strategy if a complete nonlinear helicopter model is used. For instance, Isidori et al. (2003) addressed the problem of controlling the motion of a helicopter described by a nonlinear mathematical model. To simplify the nonlinearity of the dynamics and the strong coupling effects in the model, the unavailability of simplification of the system model is applied.

Research on a helicopter carrying external underslung loads has gained great attention in the aerospace research community for the past few decades due to the re-evaluation and extension of the ADS-33 and the inherent stability problems associated with this system (ADS-33-D, 1996; Bisgaard, Bendtsen, & la Cour-Harbo, 2009; Bisgaard, Harbo, & Bendtsen, 2010; Oktay & Sultan, 2013b; Thanapalan & Zhang, 2013). Helicopters have the ability to carry large and bulky loads externally on a sling. This capability is important in many applications, ranging from lifting heavy loads to saving life. Importantly, when lives are under risk and rapid rescue operations are needed this operation is vital. The stability of the helicopter will be disturbed by the underslung load, which

is a huge obstacle for an accurate pick up or placement of the loads (Thanapalan & Wong, 2010). Thus, it is necessary to design a suitable controller which can ensure the stabilization of the helicopter system and the positioning of the underslung load under various complicated situations.

A review of reported methods for flight control law design shows that many approaches used to design the control law have involved the application of SISO techniques to each control loop individually (Manness, Gribble, & Smith, 1990). The controller design methods such as linear quadratic regular (LQR) or linear quadratic Gaussian (LQG) method, commonly referred to as LQ methods (Gribble, 1993), sliding mode control (SMC) and eigenstructure assignment are used to evaluate a multivariable control law design for helicopter flight (Thanapalan, 2015). In the case of eigenstructure assignment method, the designer attempts to find optimum pole positions (Manness et al., 1990). The main idea of SMC is to maintain the system sliding on a surface in the state space despite the uncertainties or perturbations. This is done by means of a discontinuous control law that switches between two structures, when the system passes through that surface (Edwards & Spurgeon, 1998). Many researchers use the idea of SMC to develop flight control laws, see, for example, Shtessel and Shkolnikov (2003). Sliding mode control is a technique for the design of nonlinear regulators. The first step in the two-part synthesis procedure is to specify a desired sliding subspace. This involves using regulation techniques such as LQR or eigenstructure assignment to stabilize a reduced-order system. A nonlinear controller

is then developed in the second step to asymptotically drive the system towards the regulated subsystem's so-called sliding subspace. However, designing the sliding subspace is very difficult job indeed, since there appears to be little guidance on how to design a sliding subspace, which may be limiting this design method to helicopter control applications (Thanapalan, 2015).

The method like H_∞ optimization used to design a flight control law can be considered as a frequency domain method, since this technique is similar to the design of the control law based on a transfer function matrix representation of the system and it involves frequency domain performance specifications (Luo, Liu, Yang, & Chang, 2003). For example, in the case of designing a control law for a helicopter, Yue and Postlethwaite (1990) have described the application of H_∞ optimization techniques to the determination of feedback control laws for improving the handling qualities of a combat helicopter. Quantitative feedback control technique is a control synthesis technique which involves shaping the loop transmission to meet bounds placed upon it by performance specifications in terms of desired system responses and disturbance rejection levels. A survey of the quantitative feedback control technique can be found in Horowitz (2001). The possibility of applying quantitative feedback control technique to helicopter flight control design is considered by several researches, see, for example, Snell and Stout (1998). However, due to the requirements of conservative and sequential design for each of the multivariable subsystems, it is difficult to obtain the best closed-loop performance under practical constraints. Moreover, manual-bound computation and trial and error loop shaping design procedures make it difficult to realize a stabilizing feedback control law for a helicopter system using the quantitative feedback control technique.

Model reference techniques are those synthesis procedures which can be used to design feedforward controllers. For instance, an integral inverse model following technique and controllers using the nonlinear system inverses can be considered as model reference techniques. In the case of the integral inverse model following technique a regulator is designed to minimize the error transients between the responses of the system being controlled and a model which describes the dynamics. The controller using nonlinear system inverses is essentially a procedure for the inversion of the system such that each input is linked with an output (Lane & Stengel, 1988). State estimator techniques such as Kalman filter and state observer and loop transfer techniques can be classified as output feedback methods. State estimator techniques provide a means of generating estimated state variables for feedback from available

measurements. However, the use of this method has a drawback that the use of estimated state feedback can create problems for the designer in that the resulting control laws are not, in most cases, robust to uncertainties or variations in the plant (Bryson, 1985). The use of intelligent control methods for helicopter control system design such as fuzzy control and Neural Network (NN) has also been addressed by several researchers, for example, see, Kadmiry and Driankov (2001).

Form the review of popular helicopter control methods, it is clear that considerable attention has been paid to the design of a controller to obtain a satisfactory helicopter handling quality (Thanapalan, 2015). The control problem has been tackled using different approaches ranging from linear quadratic control (Gribble, 1993), eigenstructure assignment (Garrard, Low, & Prouty, 1989; Manness et al., 1990), classical SISO techniques (Garrard et al., 1989), constrained controller design (Oktay, 2012; Oktay & Sultan, 2013a) and sliding mode control (Shtessel & Shkolnikov, 2003). Apart from the methods emphasized above, many other techniques are reported for complex modern control system design ranging from quantitative feedback theory and the singular perturbation method (Thompson, Pruyn, & Shukla, 1999).

In general there are two main approaches for control of uncertain dynamical systems, that is, deterministic and stochastic control. If the uncertainty in the system model is assumed to have statistical characterization and the desired behaviour of the system is described in a statistical sense a stochastic approach is feasible; otherwise, if structural properties and bounding conditions relating to the uncertainties are known, a deterministic approach is appropriate (Thanapalan, 2015). Deterministic feedback control of uncertain dynamical systems proposes the use of determined linear or nonlinear feedback control functions, which operate effectively over a specified magnitude range of system parameter variations and disturbances, without any online identification of the system parameters. Benefit of such an approach is that no statistical information of the system variations is required to yield the desired dynamical behaviour and, hence, the controller may have a simple structure for implementation in practical systems. However, the deterministic control design methodology requires the system state vector to be available for measurements, and the bounding knowledge of uncertainties to be known, which may put restrictions on the applications of this method.

In this paper, a nonlinear deterministic controller is designed for the helicopter with an underslung external load system. The key advantage of the proposed control method is that the controller design takes the system uncertainty into account. The designed controller can give a guaranteed stability region for the systems

considered. This method should have potential for solving some problems arising in helicopter control. The paper begins by presenting a mathematical model of the system, then describing the design of a nonlinear controller with a numerical example to illustrate the applicability, accuracy and effectiveness of the proposed method.

2. System model

Considering the control of a helicopter with an underslung load, the dynamical models of both the helicopter and load have some terms which are uncertain. The uncertainties may arise from the helicopter having to carry an unknown load or the immeasurable parameters in the dynamical models. The uncertainties may also arise from computational errors of the dynamical effects such as aerodynamics. Therefore for a realistic model uncertainties must be taken into account during the controller design.

A mathematical model of the helicopter has been described in Thanapalan (2010) and an underslung load model presented in Thanapalan and Wong (2010), which are adopted in this work. Considering the two models, a mathematical model for a helicopter carrying an underslung load can be obtained. From Thanapalan (2010) and Thanapalan and Wong (2010), if the velocities and accelerations of the helicopter are considered as the inputs to the load, the combined system model will have a structure of a cascade connection of the two subsystems (Thanapalan & Zhang, 2013). Therefore, in this paper, nonlinear systems with the following format are considered.

$$\dot{x}(t) = f(x(t)) + G(x(t))\tilde{u}(t), \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $\tilde{u} \in \mathbb{R}^m$. In general mathematical models of dynamical systems are usually imprecise due to modelling errors and exogenous disturbances. Equation (1) can be considered as the nominal part of the system model and the uncertainty can be modelled as an additive perturbation to the nominal system model; more specifically, the structure of the system has the form:

$$\dot{x}(t) = f(x(t)) + G(x(t))\tilde{u}(t) + \vartheta(x(t), u(t)), \quad (2)$$

where $\vartheta(x(t), u(t))$ models the uncertainty in the system.

Now, the underslung load is considered to be suspended from a single suspension point that is subject to motion and therefore modelled as a driven spherical pendulum. The equations that describe the load dynamics

are obtained by first considering motion with reference to the longitudinal suspension angle θ_L in the x - z plane. This is then repeated for the lateral case involving ϕ_L and the y - z plane. These are then combined to obtain the model for the motion of the load. The underslung load system has six inputs, longitudinal, lateral and vertical velocities together with the corresponding accelerations of the helicopter, whilst the outputs are the longitudinal and lateral directional suspension angles. The load is subject to an isotropic aerodynamic force (proportional to the square of its airspeed) such as would be experienced by a spherical shaped load. Aerodynamic interaction with the helicopter that may occur for example due to rotor downwash has been ignored. Finally, the sling itself is assumed to be rigid and contributing zero aerodynamic force of its own. With these assumptions, the equations governing the combined system can be derived.

So, using the kind of nonlinear system structure described above, the longitudinal motion of the helicopter with an underslung load combined system described by load suspension angle θ_L in the x - z plane with respect to the z -axis, the pitch angle θ and pitch rate q together with translation motion components u, w can be written as follows:

$$\begin{aligned} \dot{\bar{\theta}}_L(t) = & f_1(\bar{\theta}_L(t)) + G_1(\bar{\theta}_L(t))[p(x_H(t)) + q(\bar{\theta}_L(t), x_H(t))] \\ & + H(t, \bar{\theta}_L, x_H(t)), \end{aligned} \quad (3a)$$

$$\dot{x}_H(t) = f_2(\bar{\theta}_L(t), x_H(t)) + G_2\tilde{u}(t), \quad (3b)$$

where

$$\begin{aligned} \bar{\theta}_L(t) = & [\theta_{L1} \quad \theta_{L2} \quad \theta \quad \bar{q}]^T, \quad x_H = [u \quad w]^T, \\ \tilde{u}(t) = & [\theta_{1s} \quad \theta_0]^T, \end{aligned}$$

and

$$p(x_H) = [u^2 + \kappa_1 u \quad w^2 + \kappa_2 w]^T.$$

$$f_1(\bar{\theta}_L(t)) = \begin{bmatrix} \theta_{L2} \\ \frac{-g}{l_x} \sin \theta_{L1} + \frac{k_D l_x}{M_L} (\text{sign}(\dot{X}_L) \cos^3 \theta_{L1} \\ + \text{sign}(\dot{Z}_L) \sin^3 \theta_{L1}) \theta_{L2}^2 - k_\theta \theta_{L2} \\ \bar{q} \\ (X_{21}\theta + X_{22}\bar{q}) \end{bmatrix},$$

$$G_1(\bar{\theta}_L(t)) = \begin{bmatrix} 0 & 0 \\ \frac{k_D \text{sign}(\dot{X}_L) \cos \theta_{L1}}{M_L l_x} & \frac{k_D \text{sign}(\dot{Z}_L) \sin \theta_{L1}}{M_L l_x} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{aligned}
q(\bar{\theta}_L(t), x_H(t)) &= \begin{bmatrix} 0 \\ \frac{\cos \theta_{L1}}{I_x} \dot{u} + \frac{\sin \theta_{L1}}{I_x} \dot{w} \\ -\frac{2k_D}{M_L} (\text{sign}(\dot{X}_L) \cos^2 \theta_{L1} u \\ + \text{sign}(\dot{Z}_L) \sin^2 \theta_{L1} w) \theta_{L2} \\ -\frac{\kappa_1 k_D \text{sign}(\dot{X}_L) \cos \theta_{L1} u}{M_L I_x} \\ -\frac{\kappa_2 k_D \text{sign}(\dot{X}_L) \cos \theta_{L1} w}{M_L I_x} \end{bmatrix}, \\
H(t, \bar{\theta}_L(t), x_H(t)) &= \begin{bmatrix} 0 \\ 0 \\ (a_{11}u + a_{12}w) \\ X_{23}u + X_{24}w \end{bmatrix}, \\
f_2(\bar{\theta}_L(t), x_H(t)) &= \begin{bmatrix} (X_{31}\theta + X_{32}\bar{q} + X_{33}u + X_{34}w) \\ (X_{41}\theta + X_{42}\bar{q} + X_{43}u + X_{44}w) \end{bmatrix}, \\
G_2 &= \begin{bmatrix} X_{\theta_{1s}} & X_{\theta_0} \\ Z_{\theta_{1s}} & Z_{\theta_0} \end{bmatrix}.
\end{aligned}$$

It is assumed that the longitudinal motion is primarily controlled by longitudinal cyclic commands (θ_{1s}) and main rotor collective θ_0 .

Similarly, for the lateral motion the combined system described by load suspension angle ϕ_L in the y - z plane with respect to the z -axis, the roll angle φ and roll rate p together with the translation motion components v, w can be written as follows:

$$\dot{\bar{\varphi}}_L(t) = f_1(\bar{\varphi}_L(t)) + G_1(\bar{\varphi}_L(t))[p(x_H(t)) + q(\bar{\varphi}_L(t), x_H(t))] + H(t, \bar{\varphi}_L, x_H(t)), \quad (4a)$$

$$\dot{x}_H(t) = f_2(\bar{\varphi}_L(t), x_H(t)) + G_2 \hat{u}(t), \quad (4b)$$

where

$$\begin{aligned}
\dot{\bar{\varphi}}_L(t) &= [\dot{\varphi}_{L1} \dot{\varphi}_{L2} \dot{\bar{p}}]^T, p(x_H(t)) = [v^2 w^2]^T, \\
f_1(\bar{\varphi}_L(t)) &= \begin{bmatrix} \frac{-g}{I_y} \sin \varphi_{L1} + \frac{k_D I_y}{M_L} (\text{sign}(\dot{Y}_L) \cos^3 \varphi_{L1} \\ + \sin(\dot{Z}_L) \sin^3 \varphi_{L1}) \varphi_{L2}^2 - k_\varphi \varphi_{L2} \\ \bar{p} \\ (-b_{11}g\phi + (L_p - b_{11}(Y_p + w_e) \\ - b_{12}(Z_p - v_e))\bar{p}) \end{bmatrix}, \\
G_1(\bar{\varphi}_L(t)) &= \begin{bmatrix} 0 & 0 \\ \left(\frac{k_D \text{sign}(\dot{Z}_L) \cos \varphi_{L1}}{M_L I_y} \right) & \left(\frac{k_D \text{sign}(\dot{Z}_L) \sin \varphi_{L1}}{M_L I_y} \right) \\ 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
q(\bar{\varphi}_L(t), x_H(t)) &= \begin{bmatrix} 0 \\ \frac{\cos \varphi_{L1}}{I_y} \dot{v} + \frac{\sin \varphi_{L1}}{I_y} \dot{w} - \frac{2k_D}{M_L} (\text{sign}(\dot{Y}_L) \cos^2 \varphi_{L1} v \\ + \text{sign}(\dot{Z}_L) \sin^2 \varphi_{L1} w) \varphi_{L2} \end{bmatrix},
\end{aligned}$$

$$H(t, \bar{\varphi}_L(t), x_H(t)) = \begin{bmatrix} 0 \\ 0 \\ (\dot{\phi} - \bar{p}) \\ (Y_{23}v + Y_{24}w) \end{bmatrix},$$

$$\hat{u}(t) = [\theta_{1c} \quad \theta_{0T}]^T,$$

$$f_2(\bar{\varphi}_L(t), x_H(t)) = \begin{bmatrix} (Y_{31}\phi + Y_{32}\bar{p} + Y_{33}v + Y_{34}w) \\ (Y_{42}\bar{p} + Y_{43}v + Y_{44}w) \end{bmatrix},$$

$$G_2 = \begin{bmatrix} Y_{\theta_{1c}} & Y_{\theta_{0T}} \\ Z_{\theta_{1c}} & Z_{\theta_{0T}} \end{bmatrix}.$$

It is assumed that the lateral motion is primarily controlled by lateral cyclic commands (θ_{1c}) and the tail rotor collective θ_{0T} .

It is noted that the helicopter with the underslung load system modelled by Equations (3) and (4) is considered to have two main parts, that is, known and unknown (or partly known). The known terms formed the nominal part of the system model. The unknown or partly known part is considered as the uncertainty to the system. The whole system is then modelled by a nominal part with the addition of uncertainty. In fact, the known elements in the subsystem (3a) and (4a) are characterized by the prescribed triple (f_1, G_1, p) and it is desired that the nominal part of the system is stable.

3. Controller design

The goal is to develop a control law to ensure that it can stabilize the helicopter with the underslung load system modelled by Equations (3) and (4), in a real environment with uncertainties. Firstly, the condition for stability needs to be identified, in order to design a controller; Thanapalan (2016) presents the stability analysis conducted to ensure the stabilization of the helicopter system and the positioning of the underslung load at hover condition. Stability analysis and numerical results proved that if the desired condition for stability is met, then it is possible to locate the load at the specified position or its neighbourhood (Thanapalan, 2016). In this paper, a nonlinear deterministic controller is designed to ensure the stability and control of the system. The purpose of the controller is to position the load at or as close as possible to a specified location. A deterministic feedback control is developed which can be continuous and discontinuous.

Let

$$\bar{\theta}_L \rightarrow h(\bar{\theta}_L),$$

where

$$h(\bar{\theta}_L) = [h_1(\bar{\theta}_L), h_2(\bar{\theta}_L)]^T \text{ be defined by}$$

$$h_i(\bar{\theta}_L) = -(1 - \alpha_1 - \beta_1)^{-1} \gamma_1 (L_{g_i} v_1)(\bar{\theta}_L), (i = 1, 2), \quad (5)$$

where γ_1 is a positive design parameter.

Suppose $p(x_H(t)) = h(\bar{\theta}_L(t))$ (where $x_H(t) = (p^{-1} \circ h)(\bar{\theta}_L)$) is considered as a feedback control for the first subsystem (3a) and using the Lyapunov function for the first subsystem v_1 as

$$v_1(\bar{\theta}_L) = \frac{1}{2} [\varsigma_1 \theta_{L_1}^2 + \varsigma_2 \theta_{L_2}^2 + (\varsigma_3 \theta - \varsigma_4 \bar{q})^2 + \varsigma_5 \bar{q}^2], \quad (6)$$

where ς_i ($i = 1, 2, 3, 4, 5$) are design parameters to be determined. Then

$$\begin{aligned} \dot{v}_1(\bar{\theta}_L) &= \langle \nabla v_1(\bar{\theta}_L), f_1(\bar{\theta}_L) + G_1(\bar{\theta}_L(t)) [p(x_H(t)) \\ &\quad + q(\bar{\theta}_L(t), x_H(t))] + H(t, \bar{\theta}_L, x_H(t)) \rangle \\ &= (L_{f_1} v_1)(\bar{\theta}_L) + \langle \nabla v_1(\bar{\theta}_L), G_1(\bar{\theta}_L(t)) [p(x_H(t)) \\ &\quad + q(\bar{\theta}_L(t), x_H(t))] + H(t, \bar{\theta}_L, x_H(t)) \rangle \end{aligned}$$

by defining

$$\begin{aligned} \Psi_1(\bar{\theta}_L) &= k_\theta \varsigma_2 \theta_{L_2}^2 + [\varsigma_3 \varsigma_4 - (\varsigma_4^2 + \varsigma_5) X_{21}] \bar{q}^2 \\ &\quad + \varsigma_3 \varsigma_4 X_{21} \theta^2. \end{aligned}$$

and

$$\Psi_2(\bar{\theta}_L) = [\varsigma_3 \varsigma_4 - (\varsigma_4^2 + \varsigma_5) X_{21}] \bar{q}^2 + \varsigma_3 \varsigma_4 X_{21} \theta^2,$$

the following lemma can be derived.

Lemma 1: Defining a Lyapunov function (6) and choosing the design parameters to satisfy $\varsigma_1 = (2g/l_x \pi) \varsigma_2$, $[\varsigma_3 \varsigma_4 - (\varsigma_4^2 + \varsigma_5) X_{21}] > 0$, and $[\varsigma_3^2 - \varsigma_3 \varsigma_4 X_{22} + (\varsigma_4^2 + \varsigma_5) X_{21}] \geq 0$, then within the region specified by

$$-[\varsigma_3 \varsigma_4 - (\varsigma_4^2 + \varsigma_5) X_{21}] \bar{q}^2 < \varsigma_3 \varsigma_4 X_{21} \theta^2$$

- (1) $v_1(0) = 0$ and $v_1(\bar{\theta}_L) > 0, \forall \bar{\theta}_L \neq 0$
- (2) $v_1(\bar{\theta}_L) \rightarrow \infty$ as $\|\bar{\theta}_L\| \rightarrow \infty$
- (3) $(L_{f_1} v_1)(\bar{\theta}_L) \leq -\Psi_1(\bar{\theta}_L), \forall \bar{\theta}_L$ around the hover condition or $(L_{f_1} v_1)(\bar{\theta}_L) \leq -\Psi_2(\bar{\theta}_L), \forall \bar{\theta}_L$ If the hinge friction is big enough to satisfy the following:

$$\begin{aligned} k_\theta &> \frac{k_D l_x}{M_L} |\max(\theta_{L_2}) (\text{sign}(\dot{X}_L) \cos^3 \theta_{L_1} \\ &\quad + \text{sign}(\dot{Z}_L) \sin^3 \theta_{L_1})|. \end{aligned} \quad (7)$$

Both functions Ψ_1 and Ψ_2 are non-negative.

Recalling the control term $p(x_H(t))$ in the first subsystem of Equation (3a), it can be seen that $[(Dp)(x_H)]^{-1}$ exists for all $x_H(t)$.

Lemma 2: The uncertainties are bounded and satisfy

$$\|q(\bar{\theta}_L, x_H)\| \leq \frac{k_\theta}{l_x} |\theta_{L_2}| + \alpha_1 \|p(x_H)\| + \alpha_2(t) \quad (8)$$

and

$$\|H(t, \bar{\theta}_L, x_H)\| \leq \hat{\theta}^{-1}(\bar{\theta}_L) \beta_1 \sum_{i=1}^2 (|(L_{g_i} v_1)(\bar{\theta}_L)|) |p_i(x_H)| \quad (9)$$

with $\alpha_1, \beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$, for $\alpha_1 = (k_D/M_L)(2 + (\kappa/l_x))$ and $\beta_1 = (M_L l_x/k_D \varsigma_2) \|\bar{A}\|$. then

$$\begin{aligned} \dot{v}_1(\bar{\theta}_L) &= \langle (\nabla v_1)(\bar{\theta}_L), \dot{\bar{\theta}}_L \rangle \\ &= \left\langle (\nabla v_1)(\bar{\theta}_L), f_1(\bar{\theta}_L) + \sum_{i=1}^2 g_i(\bar{\theta}_L)(h_i(\bar{\theta}_L) \right. \\ &\quad \left. + q_i(p^{-1} \circ h)(\bar{\theta}_L)) + H(t, \bar{\theta}_L, (p^{-1} \circ h)) \right\rangle \\ &\leq -\bar{\Psi}(\bar{\theta}_L) - (\gamma_1 - \mu_1) |(L_{g_2} v_1)(\bar{\theta}_L)|^2 \\ &\quad + \alpha_2(t) |(L_{g_2} v_1)(\bar{\theta}_L)| \\ &\leq -\bar{\Psi}(\bar{\theta}_L) - \left[\sqrt{\gamma_1 - \mu_1} |(L_{g_2} v_1)(\bar{\theta}_L)| \right. \\ &\quad \left. - \frac{\alpha_2(t)}{2\sqrt{\gamma_1 - \mu_1}} \right]^2 + \frac{\alpha_2^2(t)}{4(\gamma_1 - \mu_1)}, \end{aligned}$$

where $\bar{\Psi}$ represents either Ψ_1 or Ψ_2 . That is, the solutions for the first subsystem will tend to a compact set if $p(x_H(t)) = h(\bar{\theta}_L(t))$. Therefore, the next step of the controller design is to find a control law which will drive the second subsystem state variables to lead $p(x_H(t)) \rightarrow h(\bar{\theta}_L(t))$ while t increases. To measure how close $p(x_H(t))$ and $h(\bar{\theta}_L(t))$, a new state variable vector is defined by

$$e(t) := ((p \circ x_H) - (h \circ \theta_L))(t). \quad (10)$$

So we have

$$\dot{e}(t) = (Dp)(x_H) \dot{x}_H(t) - (Dh)(\bar{\theta}_L) \dot{\bar{\theta}}_L(t)$$

with the initial condition

$$e(0) = e^0 := p(x_H^0) - h(\theta_L^0).$$

Substituting the original system equations, we can obtain

$$\begin{aligned} \dot{e}(t) &= \Lambda(t, \bar{\theta}_L, e(t)) \dot{e}(t) \\ &= (Dp)(\tilde{p})(f_2(\bar{\theta}_L, \tilde{p}) + G_2 F(t, \bar{\theta}_L, \tilde{p})) \\ &\quad - (Dh)(\bar{\theta}_L)(f_1(\bar{\theta}_L) + G_1(\bar{\theta}_L)(e(t) + h(\bar{\theta}_L) \\ &\quad + q(\bar{\theta}_L(t), \tilde{p}))(H(t, \bar{\theta}_L, \tilde{p})), \end{aligned}$$

where $\tilde{p} = p^{-1} \circ (e(t) + h(\bar{\theta}_L))$.

Then the system with the newly defined state variable of $(\bar{\theta}_L, e)$ can be modelled by

$$\begin{aligned} \begin{bmatrix} \dot{\bar{\theta}}_L \\ \dot{e} \end{bmatrix} &= \begin{bmatrix} f_1(\bar{\theta}_L) + G_1(\bar{\theta}_L)(e + h(\bar{\theta}_L)) \\ +q(\bar{\theta}_L, \bar{p}) + H(t, \bar{\theta}_L, \bar{p}) \\ \Lambda(t, \bar{\theta}_L, e) \end{bmatrix} \\ &= \Gamma(t, \bar{\theta}_L, e). \end{aligned} \quad (11)$$

Choose A_2, Q_2 such that $\sigma(A_2) \subset C^-$ and Q_2 is a symmetric positive-definite matrix, i.e. $Q_2 > 0$. Let $P_2 > 0$ denote the unique symmetric solution of the Lyapunov equation

$$P_2 A_2 + A_2^T P_2 + Q_2 = 0. \quad (12)$$

The continuous state feedback control $F(t, \bar{\theta}_L, x_H)$ is designed to have the following structure:

$$\begin{aligned} (t, \bar{\theta}_L, x_H) &\mapsto F(t, \bar{\theta}_L, x_H) \\ &= G_2^{-1}(u_f(\bar{\theta}_L, x_H) + u_{gain}(t, \bar{\theta}_L, x_H)), \end{aligned} \quad (13)$$

where

$$\begin{aligned} u_f(\bar{\theta}_L, x_H) &:= -f_2(\bar{\theta}_L, x_H) + [(Dp)(x_H)]^{-1}(A_2(p(x_H) \\ &\quad - h(\bar{\theta}_L)) + (Dh)(\bar{\theta}_L)(f_1(\bar{\theta}_L) + G_1(\bar{\theta}_L)p(x_H))) \end{aligned}$$

and

$$\begin{aligned} u_{gain}(t, \bar{\theta}_L, x_H) &:= -\rho_C(t, \bar{\theta}_L, x_H)\Pi(\rho_C(t, \bar{\theta}_L, x_H) \\ &\quad \times [(Dp)(x_H)]^T P_2(p(x_H) - h(\bar{\theta}_L))). \end{aligned}$$

In u_N , ρ_C is defined as any continuous function satisfying the following inequality:

$$\begin{aligned} \rho_C(t, \bar{\theta}_L, x_H) &\geq \sum_{i=1}^2 (\mu_1 |(L_{g_i} v_1)(\bar{\theta}_L)| + \alpha_1 |p_i(x_H)| + \alpha_2(t)) \\ &\quad \times \|[(Dp)(x_H)]^{-1}(Dh)(\bar{\theta}_L)g_i\| \\ &\quad + \max(\|[(Dp)(x_H)]^{-1}(Dh)(\bar{\theta}_L)\| \|H(\bar{\theta}_L, x_H)\|) \end{aligned} \quad (14)$$

and

$$Z \rightarrow \Pi(Z) =: \begin{cases} \|Z\|^{-1}Z & \text{if } \|Z\| > \eta, \\ Z/\eta & \text{otherwise,} \end{cases} \quad (15)$$

where η is a positive design parameter to be determined. The feedback control in Equation (13) can be considered to have two parts. The first part is a nonlinear feedback which is to stabilize the known part of the system and the second part is a variable gain which is to address the uncertainty in the system.

For system (11), a Lyapunov function is chosen as

$$\tilde{v} \left(\begin{bmatrix} \bar{\theta}_L \\ e \end{bmatrix} \right) = v_1(\bar{\theta}_L) + \xi v_2(e), \quad (16)$$

where $\xi > 0$ is a design parameter to be specified and $v_2(e) =: (1/2) \langle e, P_2 e \rangle$. Then for $\|\rho_C(t, \bar{\theta}_L, x_H)[(Dp)(x_H)]^T P_2(p(x_H) - h(\bar{\theta}_L))\| < \eta$, we have

$$\begin{aligned} \tilde{v} &= \langle \nabla v_1(\bar{\theta}_L), \dot{\bar{\theta}}_L \rangle + \xi \langle \nabla v_2(e), \dot{e} \rangle \\ &\leq -\bar{\Psi}(\bar{\theta}_L) - (1 + \alpha_1 + \beta_1) \sum_{i=1}^2 \left(\frac{1}{2} |(L_{g_i} v_1)(\bar{\theta}_L)| - |e_i| \right)^2 \\ &\quad - \left[\mu_1(\gamma_1 - 1) - \frac{1}{4}(1 + \alpha_1 + \beta_1) \right] \sum_{i=1}^2 \left(|(L_{g_i} v_1)(\bar{\theta}_L)| \right. \\ &\quad \left. - \frac{2\alpha_2}{4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)} |e_i| \right)^2 \\ &\quad - \left[\frac{1}{2} \xi \sigma_{\min}(Q_2) - \frac{1}{4}(1 + \alpha_1 + \beta_1) \right] \|e\|^2 \\ &\quad + 2\xi\eta + \frac{\alpha_2^2}{[4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)]}, \end{aligned}$$

for $\|\rho_C(t, \bar{\theta}_L, x_H)[(Dp)(x_H)]^T P_2(p(x_H) - h(\bar{\theta}_L))\| \geq \eta$, we have

$$\begin{aligned} \tilde{v} &\leq -\bar{\Psi}(\bar{\theta}_L) - (1 + \alpha_1 + \beta_1) \sum_{i=1}^2 \left(\frac{1}{2} |(L_{g_i} v_1)(\bar{\theta}_L)| - |e_i| \right)^2 \\ &\quad - [\mu_1(\gamma_1 - 1) - \frac{1}{4}(1 + \alpha_1 + \beta_1)] \sum_{i=1}^2 \left(|(L_{g_i} v_1)(\bar{\theta}_L)| \right. \\ &\quad \left. - \frac{2\alpha_2}{4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)} |e_i| \right)^2 \\ &\quad - \left[\frac{1}{2} \xi \sigma_{\min}(Q_2) - \frac{1}{4}(1 + \alpha_1 + \beta_1) \right] \|e\|^2 \\ &\quad + \frac{\alpha_2^2}{[4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)]}. \end{aligned}$$

For both cases, the following is true:

$$\begin{aligned} \tilde{v} &\leq -\bar{\Psi}(\bar{\theta}_L) - (1 + \alpha_1 + \beta_1) \sum_{i=1}^2 \left(\frac{1}{2} |(L_{g_i} v_1)(\bar{\theta}_L)| - |e_i| \right)^2 \\ &\quad - [\mu_1(\gamma_1 - 1) - \frac{1}{4}(1 + \alpha_1 + \beta_1)] \sum_{i=1}^2 \left(|(L_{g_i} v_1)(\bar{\theta}_L)| \right. \\ &\quad \left. - \frac{2\alpha_2}{4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)} |e_i| \right)^2 \\ &\quad - \left[\frac{1}{2} \xi \sigma_{\min}(Q_2) - \frac{1}{4}(1 + \alpha_1 + \beta_1) \right] \|e\|^2 \\ &\quad + 2\xi\eta + \frac{\alpha_2^2}{[4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)]}. \end{aligned}$$

Therefore

$$\dot{v} \leq - \left\langle \begin{bmatrix} \bar{\Psi}^{1/2}(\bar{\theta}_L) \\ \|e(t)\| \end{bmatrix}, T_1 \begin{bmatrix} \bar{\Psi}^{1/2}(\bar{\theta}_L) \\ \|e(t)\| \end{bmatrix} \right\rangle + \frac{\alpha_2^2}{[4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)]} + 2\eta\xi, \quad (17)$$

where

$$T_1 := \begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2}\xi\sigma_{\min}(Q_2) - \frac{1}{4}(1 + \alpha_1 + \beta_1)) \end{bmatrix}.$$

Choosing $4\mu_1(\gamma_1 - 1) > (1 + \alpha_1 + \beta_1)$, $\gamma_1 > 1 \geq ((1 + \alpha_1 + \beta_1)/2\sigma_{\min}(Q_2))$ ensures that $\sigma_{\min}(T_1) = 1$ and guarantees that

$$\dot{v} \left(\begin{bmatrix} \bar{\theta}_L(t) \\ e(t) \end{bmatrix} \right) \leq 0 \quad a.e., \quad \forall \begin{bmatrix} \bar{\theta}_L(t) \\ e(t) \end{bmatrix} \in \mathbb{R}^4 \setminus \lambda(k^\eta),$$

where

$$\lambda(k^\eta) := \left\{ \begin{bmatrix} \bar{\theta}_L(t) \\ e(t) \end{bmatrix} : \left\| \begin{bmatrix} \bar{\Psi}^{1/2}(\bar{\theta}_L) \\ e(t) \end{bmatrix} \right\| \leq k^\eta; k^\eta := \frac{\alpha_2^2}{[4\mu_1(\gamma_1 - 1) - (1 + \alpha_1 + \beta_1)]} + 2\eta\xi \right\}.$$

From the above analysis, the following theorem results:

Theorem 1: Following Lemmas 1 and 2 with the feedback control defined in Equation (13), if the design parameters satisfy $4\mu_1(\gamma_1 - 1) > (1 + \alpha_1 + \beta_1)$, $\gamma_1 > 1$, and $\xi \geq ((1 + \alpha_1 + \beta_1)/2\sigma_{\min}(Q_2))$, the compact set $\lambda(k^\eta)$ is globally asymptotically stable for the helicopter dynamic system.

Theorem 1 indicates that the feedback control (13) can position the load at the specified location or to a small region around the specified location. The size of the region can be reduced by choosing proper design parameters.

The procedure for the development of a control law for the lateral motion is similar to the longitudinal motion. Therefore, the stabilizing feedback control law for lateral motion can be written with the appropriate parameter:

$$\begin{aligned} (t, \bar{\varphi}_L, x_H) &\mapsto F(t, \bar{\varphi}_L, x_H) \\ &= G_2^{-1}(u_L(\bar{\varphi}_L, x_H) + u_N(t, \bar{\varphi}_L, x_H)), \end{aligned} \quad (18)$$

whereas

$$\begin{aligned} u_L(\bar{\varphi}_L, x_H) &:= -f_2(\bar{\varphi}_L, x_H) + [(Dp)(x_H)]^{-1} \\ &\quad \times (A_2(p(x_H) - h(\bar{\varphi}_L)) + (Dh)(\bar{\varphi}_L)(f_1(\bar{\varphi}_L) \\ &\quad + G_1(\bar{\varphi}_L)p(x_H))) \end{aligned}$$

$$\begin{aligned} u_N(t, \bar{\varphi}_L, x_H) &= -\rho_C(t, \bar{\varphi}_L, x_H)\Pi(\rho_C(t, \bar{\varphi}_L, x_H)[(Dp)(x_H)]^T \\ &\quad \times P_2(p(x_H) - h(\bar{\varphi}_L))), \end{aligned}$$

where ρ_C is a gain function which is introduced to address the uncertainties in the system.

4. Numerical analysis

In this section, the feedback control developed in Section 3 is applied to a helicopter model in its linearized model with an underslung load. Only the longitudinal case is studied here. The load model adopted in this paper is the same as the model in Thanapalan and Wong (2010), Thanapalan and Zhang (2013) which has a weight $M_L = 1000/b$ with a $l_x = 15ft$ sling length, $k_D = 75$ and $k_\theta = 2.5$ and in this example, a linear model of helicopter (Garrard et al., 1989) is adopted, which has the system parameters as below:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{p} \\ \dot{q} \\ \dot{r} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -0.0199 & -0.0058 & -0.0058 & -0.0151 \\ -0.0452 & -0.526 & -0.0061 & -0.026 \\ -0.0788 & -0.0747 & -0.3803 & 0.0008 \\ 0.4557 & -2.5943 & -0.1787 & -2.9979 \\ 0.3688 & 0.1931 & -0.1753 & 0.071 \\ 1.0939 & 0.731 & -0.0358 & 0.4058 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \\ \phi \\ \theta \end{bmatrix} + \begin{bmatrix} 0.0232 & 0.0006 & 0 & -0.06652 \\ -0.0155 & 0.0148 & 0.6648 & -0.0003 \\ -0.0048 & 0.042 & 0.0228 & 0.0102 \\ -0.5943 & 0.4155 & 0 & 0 \\ -0.5943 & 0.0013 & 0 & 0 \\ 0.4069 & -0.4940 & 0 & 0 \\ 0.0005 & -0.0154 & 0 & 0 \\ 0.9994 & 0.0343 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_o \\ \theta_{1s} \\ \theta_{1c} \\ \theta_{0T} \end{bmatrix} + \begin{bmatrix} -0.0456 & -0.083 & 0.4735 & -0.0016 \\ -0.0369 & 0.2785 & 0.0086 & 0.3600 \\ -3.1126 & -0.0032 & 0.0076 & 0.0002 \\ -2.4241 & 20.8327 & 1.0196 & 9.1903 \\ -0.03205 & 0.02538 & -6.3329 & -0.0648 \\ 5.7889 & -2.6208 & 2.3832 & -11.0904 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_o \\ \theta_{1s} \\ \theta_{1c} \\ \theta_{0T} \end{bmatrix}.$$

If only the longitudinal motion is considered, then we have

$$\begin{bmatrix} \dot{\theta} \\ \dot{q} \\ \dot{u} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0 & 0.9994 & 0 & 0 \\ 0 & -0.5943 & 0.3688 & -0.1753 \\ -0.06652 & 0.0232 & -0.0199 & -0.0058 \\ 0.0102 & -0.0048 & -0.0788 & -0.3803 \end{bmatrix} \begin{bmatrix} \theta \\ q \\ u \\ w \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.02538 & -0.03205 \\ -0.083 & -0.0456 \\ -0.0032 & -3.1126 \end{bmatrix} \begin{bmatrix} \theta_{1s} \\ \theta_o \end{bmatrix}.$$

From the above system parameters, the following can be obtained:

$$a_{11} = -0.30635 \quad \text{and} \quad a_{22} = 0.014785.$$

Let

$$\bar{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -0.30635 & 0.014785 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then

$$\bar{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0.30635 & -0.014785 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So the original system model can be transformed into the following format:

$$\begin{aligned} \bar{T}^{-1} \begin{bmatrix} \dot{\theta} \\ \dot{q} \\ \dot{u} \\ \dot{w} \end{bmatrix} &= \bar{T}^{-1} \begin{bmatrix} 0 & 0.9994 \\ 0 & -0.5943 \\ -0.06652 & 0.0232 \\ 0.0102 & -0.0048 \end{bmatrix} \bar{T} \bar{T}^{-1} \begin{bmatrix} \theta \\ q \\ u \\ w \end{bmatrix} \\ &+ \bar{T}^{-1} \begin{bmatrix} 0 & 0 \\ 0.02538 & -0.03205 \\ -0.083 & -0.0456 \\ -0.0032 & -3.1126 \end{bmatrix} \begin{bmatrix} \theta_{1s} \\ \theta_0 \end{bmatrix}. \end{aligned}$$

That is,

$$\begin{aligned} \begin{bmatrix} \dot{\theta} \\ \dot{q} \\ \dot{u} \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} 0 & 0.9994 & -0.3045119 & 0.014696 \\ -0.02053 & -0.58712 & 0.54373 & -0.18013 \\ -0.06652 & 0.0232 & -0.02701 & -0.0058 \\ 0.0102 & -0.0048 & -0.0788 & -0.3803 \end{bmatrix} \begin{bmatrix} \theta \\ q \\ u \\ w \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.083 & -0.0456 \\ -0.0032 & -3.1126 \end{bmatrix} \begin{bmatrix} \theta_{1s} \\ \theta_0 \end{bmatrix}. \end{aligned}$$

The system transformed has the following structure:

$$\begin{aligned} f_1(\bar{\theta}_L(t)) &= \begin{bmatrix} \theta_{L2} \\ -2.133 \sin \theta_{L1} + 1.125(\dot{X}_L) \cos^3 \theta_{L1} \\ + \text{sign}(\dot{Z}_L) \sin^3 \theta_{L1} \theta_{L2}^2 - 2.5 \theta_{L2} \\ \bar{q} \\ -0.02053 \theta + -0.58712 \bar{q} \end{bmatrix}, \\ G_1(\bar{\theta}_L(t)) &= \begin{bmatrix} 0 & 0 \\ \frac{k_D \text{sign}(\dot{X}_L) \cos \theta_{L1}}{M_L I_x} & \frac{k_D \text{sign}(\dot{X}_L) \sin \theta_{L1}}{M_L I_x} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0.005 \text{sign}(\dot{X}_L) \cos \theta_{L1} & 0.005 \text{sign}(\dot{Z}_L) \sin \theta_{L1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

$$q(\bar{\theta}_L(t), x_H(t)) = \begin{bmatrix} 0 \\ 0.0667 \cos \theta_{L1} \dot{u} + 0.0667 \sin \theta_{L1} \dot{w} \\ -0.15(\text{sign}(\dot{X}_L) \cos^2 \theta_{L1} u \\ + \text{sign}(\dot{Z}_L) \sin^2 \theta_{L1} w) \theta_{L2} \end{bmatrix},$$

$$H(t, \bar{\theta}_L(t), x_H(t)) = \begin{bmatrix} 0 \\ 0 \\ -0.3045119u + 0.014696w \\ 0.54373u - 0.18013w \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -0.083 & -0.0456 \\ -0.0032 & -3.1126 \end{bmatrix}$$

and

$$\begin{aligned} f_2(\bar{\theta}_L(t), x_H(t)) &= \begin{bmatrix} (-0.06652\theta + 0.0232\bar{q} - 0.02801u - 0.0058w) \\ (0.0102\theta - 0.0048\bar{q} - 0.0788u - 0.3803w) \end{bmatrix}. \end{aligned}$$

The next step is to check if the system satisfies all the conditions required for application of the method developed in Section 3. With the given system parameters, we can calculate the following:

$$\alpha_1 = \frac{k_D}{M_L} \left(2 + \frac{\kappa}{I_x} \right) = 0.15.$$

$$\|\bar{A}\| = \max(\sigma(\bar{A}^T \bar{A}))$$

$$\begin{aligned} &= \max \left(\sigma \left(\begin{bmatrix} -0.3045119 & 0.0149629 \\ 0.54373 & -0.180135 \end{bmatrix}^T \right. \right. \\ &\quad \left. \left. \times \begin{bmatrix} -0.3045119 & 0.0149629 \\ 0.54373 & -0.180135 \end{bmatrix} \right) \right) \\ &= \max \left(\sigma \left(\begin{bmatrix} 0.38837 & -0.10242 \\ -0.10242 & 0.03266 \end{bmatrix} \right) \right) = 0.6447, \end{aligned}$$

$$\beta_1 = \frac{M_L I_x}{k_D \varsigma_2} \|\bar{A}\| = 128.8/\varsigma_2.$$

If ς_2 is chosen to be 200, then $\beta_1 = 128.8/\varsigma_2 = 0.644$. Hence, $\alpha_1 + \beta_1 = 0.794 < 1$ is true. And also, we can obtain $\mu_1 = (k_\theta M_L / k_D \varsigma_2) = 0.1667$. From Section 3, the design parameter ς_1 can be chosen as $\varsigma_1 = (2g/I_x \pi) \varsigma_2 = 271.62$.

The following is to check if Equation (7) holds with the given parameters. For the given system parameters, we have

$$\begin{aligned} &\frac{k_D I_x}{M_L} |\max(\theta_{L2}) (\text{sign}(\dot{X}_L) \cos^3 \theta_{L1} + \text{sign}(\dot{Z}_L) \sin^3 \theta_{L1})| \\ &\leq \frac{k_D I_x}{M_L} |\max(\theta_{L2})| 2 \leq 2.25 |\max(\theta_{L2})|. \end{aligned}$$

In general, one cannot imagine that the load can swing over 45° of angle within 1 s. Therefore, it is realistic to

assume that $|\max(\theta_{L2})| \leq 1$, which leads to

$$\frac{k_D l_x}{M_L} |\max(\theta_{L2}) (\text{sign}(\dot{X}_L) \cos^3 \theta_{L1} + \text{sign}(\dot{Z}_L) \sin^3 \theta_{L1})| \leq 2.25 |\max(\theta_{L2})| < k_\theta,$$

that is, Equation (7) holds.

If the design parameters ς_3 , ς_4 , and ς_5 are chosen to be $\varsigma_3 = 1$, $\varsigma_4 = 0$, and $\varsigma_5 = 1$, the inequalities $[\varsigma_3 \varsigma_4 - (\varsigma_4^2 + \varsigma_5) X_{21}] > 0$ and $[\varsigma_3^2 - \varsigma_3 \varsigma_4 X_{22} + (\varsigma_4^2 + \varsigma_5) X_{21}] \geq 0$ hold. If γ_1 is chosen to be 3.8, $4\mu_1(\gamma_1 - 1) > (1 + \alpha_1 + \beta_1)$ and $\gamma_1 > 1$ are true. For the purpose of simplifying analysis, it is preferred to choose $A_2 = -3I$, $P_2 = (1/3)I$ so $Q_2 = 2I$. In this case, the design parameter ξ is given a value of 0.5 and $\xi \geq ((1 + \alpha_1 + \beta_1)/2\sigma_{\min}(Q_2))$ holds.

With all the above analyses, the suitable Lyapunov function for the system is

$$\tilde{v} \left(\begin{bmatrix} \bar{\theta}_L \\ e \end{bmatrix} \right) = \frac{1}{2} [271.62\theta_{L1}^2 + 200\theta_{L2}^2 + \theta^2 + \bar{q}^2] + \frac{1}{6} \langle e, e \rangle.$$

Remark 1: It is important to note that the Lyapunov function is subject to changes with respect to the size of the compact set of $\lambda(k^\eta)$.

For the chosen Lyapunov function, we have that $\Psi_1(\bar{\theta}_L) = 500\theta_{L2}^2 + 0.3045\bar{q}^2$ and $-[\varsigma_3 \varsigma_4 - (\varsigma_4^2 + \varsigma_5) X_{21}]\bar{q}^2 < \varsigma_3 \varsigma_4 X_{21}\theta^2$ holds for all situations, which implies that the solution is applicable to the whole variable range.

Based on the above system and design parameters,

$$h_1(\bar{\theta}_L) = -(1 - \alpha_1 - \beta_1)^{-1} \gamma_1 (L_{g1} v_1)(\bar{\theta}_L) = -18.447\theta_{L2}$$

and $h_2(\bar{\theta}_L) = -(1 - \alpha_1 - \beta_1)^{-1} \gamma_1 (L_{g2} v_1)(\bar{\theta}_L) = -18.447\theta_{L2}$ as well.

With $G_2 = \begin{bmatrix} -0.083 & -0.0456 \\ -0.0032 & -3.1126 \end{bmatrix}$, we have $G_2^{-1} = \begin{bmatrix} -12.055 & 0.1766 \\ 0.0124 & -0.3215 \end{bmatrix}$.

For the specified system, we can derive the following:

$$[(Dp)(x_H)]^{-1} = \begin{bmatrix} \frac{1}{2u + \kappa_1} & \\ & \frac{1}{2w + \kappa_2} \end{bmatrix}$$

and

$$A_2(p(x_H) - h(\bar{\theta}_L)) = -3 \begin{bmatrix} u^2 + \kappa_1 u + 18.447\theta_{L2} \\ w^2 + \kappa_2 w + 18.447\theta_{L2} \end{bmatrix}$$

With $(Dh)(\bar{\theta}_L) = \begin{bmatrix} 0 & -18.447 & 0 & 0 \\ 0 & -18.447 & 0 & 0 \end{bmatrix}$, we have

$$(Dh)(\bar{\theta}_L)(f_1(\bar{\theta}_L) + G_1(\bar{\theta}_L)p(x_H)) = \begin{bmatrix} 39.354 \sin \theta_{L1} + -20.753(\text{sign}(\dot{X}_L) \cos^3 \theta_{L1} + \text{sign}(\dot{Z}_L) \sin^3 \theta_{L1})\theta_{L2}^2 - 2.5\theta_{L2} \\ 39.354 \sin \theta_{L1} + -20.753(\text{sign}(\dot{X}_L) \cos^3 \theta_{L1} + \text{sign}(\dot{Z}_L) \sin^3 \theta_{L1})\theta_{L2}^2 - 2.5\theta_{L2} \end{bmatrix} - \begin{bmatrix} -18.447[0.005 \text{sign}(\dot{X}_L) \cos \theta_{L1} (u^2 + \kappa_1 u) + 0.005 \text{sign}(\dot{Z}_L) \sin \theta_{L1} (w^2 + \kappa_2 w)] \\ -18.447[0.005 \text{sign}(\dot{X}_L) \cos \theta_{L1} (u^2 + \kappa_1 u) + 0.005 \text{sign}(\dot{Z}_L) \sin \theta_{L1} (w^2 + \kappa_2 w)] \end{bmatrix}.$$

Therefore, we can obtain the first part of the feedback control by substituting all the above derived functions:

$$u_f(\bar{\theta}_L, x_H) := -f_2(\bar{\theta}_L, x_H) + [(Dp)(x_H)]^{-1} (A_2(p(x_H) - h(\bar{\theta}_L)) + (Dh)(\bar{\theta}_L)(f_1(\bar{\theta}_L) + G_1(\bar{\theta}_L)p(x_H))).$$

To derive the part addressing the uncertainties of the feedback control, the gain function needs to be examined, which is described below:

$$\begin{aligned} & \sum_{i=1}^2 (\mu_1 |(L_{g_i} v_1)(\bar{\theta}_L)| + \alpha_1 |p_i(x_H)| + \alpha_2(t)) \\ & \times \|[(Dp)(x_H)]^{-1} (Dh)(\bar{\theta}_L) g_i\| \\ & + \max(\|[(Dp)(x_H)]^{-1} (Dh)(\bar{\theta}_L)\| \|H(\bar{\theta}_L, x_H)\|) \\ & \leq \|0.3334\theta_{L2} + 0.15(u^2 + \kappa_1 u) + 0.15(w^2 + \kappa_2 w)\| 0.092 \\ & + 340.29 \sqrt{\frac{(-0.3045u + 0.0147w)^2}{+(-0.54373u - 0.1803w)^2}} \\ & \times \sqrt{\frac{1}{(2u + \kappa_1)^2} + \frac{1}{(2w + \kappa_2)^2}} \\ & \leq \{0.03334|\theta_{L2}| + 0.015(u^2 + w^2) + 64.06(|u| + |w|)\} \\ & \times \left| \frac{u + w + \kappa}{(2u + \kappa_1)(2w + \kappa_2)} \right|. \end{aligned}$$

So we can choose

$$\rho_C(t, \bar{\theta}_L, x_H) = \{0.03334|\theta_{L2}| + 0.015(u^2 + w^2) + 64.06(|u| + |w|)\} \left| \frac{u + w + \kappa}{(2u + \kappa_1)(2w + \kappa_2)} \right|$$

$[(Dp)(x_H)]^T P_2(p(x_H) - h(\bar{\theta}_L))$ can be obtained as follows:

$$\begin{aligned} & [(Dp)(x_H)]^T P(p(x_H) - h(\bar{\theta}_L))_2 \\ & = \frac{1}{3} \begin{bmatrix} (2u + \kappa_1)[u^2 + \kappa_1 u + 18.447\theta_{L2}] \\ (2w + \kappa_1)[w^2 + \kappa_1 w + 18.447\theta_{L2}] \end{bmatrix}. \end{aligned}$$

From all the above analyses, $u_{gain}(t, \bar{\theta}_L, x_H)$ can be derived:

$$u_{gain}(t, \bar{\theta}_L, x_H) = -\rho_C(t, \bar{\theta}_L, x_H) \Pi(\rho_C(t, \bar{\theta}_L, x_H) \times [(Dp)(x_H)]^T P_2 [p(x_H) - h(\bar{\theta}_L)]).$$

Therefore, all the terms in the feedback control are obtained and the feedback control is $F(t, \bar{\theta}_L, x_H) = G_2^{-1} [u_f(\bar{\theta}_L, x_H) + u_{gain}(t, \bar{\theta}_L, x_H)]$.

5. Discussion and concluding remarks

In this paper, a generalized state feedback control is presented, which has been proved to be able to locate the load at the specified position or its neighbourhood. Furthermore, numerical analysis is presented with an illustration example to show the applicability, accuracy and effectiveness of the proposed method. The advantages of the method are (1) the system uncertainties are taken into account prior to the controller design which leads to a robust feedback control; (2) the method results in a guaranteed load positioning accuracy which depends on the design parameters; (3) the controller can be further simplified with the analysis to an individual helicopter system. The main disadvantage is that the controller requires the full state feedback which may lead to implementation difficulties.

Only the longitudinal case is discussed for the controller design but the lateral case can be followed easily. In the analysis, a linearized helicopter model is adopted which has simplified the design procedure but the quality of the feedback control is limited by the accuracy of the system model. Due to the complexity of nonlinear helicopter model, the compromise between the control quality and simplicity has to be made in most situations.

Notation

L, M, N	overall helicopter rolling, pitching and yawing moments
p, q, r	helicopter roll, pitch and yaw rates about body reference axes
X, Y, Z	overall helicopter force components
φ, θ, ψ	roll, pitch and yaw angles
u, v, w	helicopter velocity components at centre of gravity
θ_0	main rotor collective
θ_{1s}	longitudinal cyclic commands
θ_{1c}	lateral cyclic commands
θ_{0T}	tail rotor collective
X_0, Y_0, Z_0	location of the suspension point with respect to earth referenced x, y and z directions

$\dot{X}_0, \dot{Y}_0, \dot{Z}_0$	the helicopter velocity in the x, y and z directions
$\ddot{X}_0, \ddot{Y}_0, \ddot{Z}_0$	the helicopter acceleration in the x, y and z directions
θ_L	load suspension angle in the x - z plane with respect to the z -axis
φ_L	load suspension angle in the y - z plane with respect to the z -axis
M_L	mass of the suspended load
g	acceleration due to gravity
l	sling length
k_D	aerodynamic drag force constant ($k_D = \frac{1}{2} \rho S C_D$)
S	the load area presented to the airflow
C_D	drag coefficient for the load

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